8.1. Almost complex structure

An almost complex structure J on a manifold M^m is a $(1,1)$-tensor field with the following property: if for every $p \in M$ we denote by $J_p : T_pM \to T_pM$ the linear map associated with J (recall Theorem T.3), then

$$J_p \circ J_p = -\mathrm{id}_{T_pM}.$$

Prove that every complex manifold admits an almost complex structure.

Hint: Composed with the differential of a complex chart $\varphi : U \to \varphi(U) \subset \mathbb{C}^n$, J_p amounts to the multiplication by i.

8.2. Kähler manifolds

Let M be a complex manifold with an almost complex structure $J \in \Gamma(T_{1,1}M)$ (as in Exercise 1). Suppose that M is endowed with an hermitian metric, that is, $g_p(J_pv, J_pw) = g_p(v,w)$ for all $p \in M$ and $v, w \in T_pM$. Show that

$$\omega(X,Y) \coloneqq g(X, JY)$$

defines a 2-form $\omega \in \Omega^2(M)$, which is closed if and only if J is parallel (i.e. $DJ = D^{1,1}J \equiv 0$).

8.3. Cut locus Prove the second and third assertion of Proposition 6.5 in the lecture notes.